

Quantum Deformation of the Non-Polynomial Interaction

$$V(r) = r^2 + \lambda r^2/(1 + g r^2)$$

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A general scheme for studying q -deformed quasi-exactly solvable problems has been developed using the partial algebraization method. Exact solutions of the deformed non-polynomial interaction generated by the potential $V(r) = r^2 + \lambda r^2/(1 + g r^2)$ are obtained.

1. Introduction

Of late, quantum deformation of Lie algebras, also called quantum groups [1–6], has attracted much attention. Apart from mathematical curiosity, quantum groups play an important role in conformal field theory [7], Yang Baxter equation [8], inverse scattering theory [9], geometric quantization [10], etc. However, application of quantum groups in quantum mechanics both relativistic and non-relativistic, is a comparatively new subject. Biedenharn [11] and Macfarlane [12] have shown how to realize the $SU_q(2)$ algebra using deformed harmonic oscillators. $SU_q(2)$ is also used to determine the effect of deformation on physical observations. Quantum deformation of wave equations is another line of investigation, whereby one can obtain the deformation effect in eigenvalues and other physical observables [13–18].

However, in almost all the cases the deformed wave equation was nonlinear, and only a first order (in terms of q) approximation can be obtained in some cases. In this paper we show how exact solutions of deformed wave equations can be obtained in case of so called “quasi exactly solvable potentials” [19–21], where one can apply the partial algebraization technique using finite dimensional representations of $SU(2)$. We discuss the formalism upto $j = 1$ representation of $SU(2)$. $j = 1$ is the most simple and elegant example of an exactly solvable deformed Schrödinger equation when the coupling parameters satisfy a certain constraint relation. For a particular example we take the non-polynomial potential given by

$$V(r) = r^2 + \frac{\lambda r^2}{1 + g r^2}.$$

This potential has been studied by a number of authors [22–24]. The plan of the paper is as follows.

In Sect. 2 we briefly discuss the essential features of the partial algebraization technique. In this section the commutation relations of $SU_q(2)$ generators are also given. In Sect. 3 we use a modified version of the partial algebraization method to find exact eigenvalues of the deformed non-polynomial potential when the coupling parameters λ, g satisfy some constraint equations. Since for $j = 0$ and $j = 1/2$ no deformation occurs, we discuss in detail the case $j = 1$. Also the eigenvalues are obtained for small $\tau(\ln q)$, where q is the deformation parameter, to show that when $\tau \rightarrow 0$ we get back the non-deformed results obtained by previous authors using supersymmetry and other considerations [25, 26].

2. Partial Algebraization Technique

Given a Schrödinger equation

$$H\psi(x) = E\psi(x), \quad (2.1)$$

we perform an imaginary gauge transformation on the wave function $\psi(x)$ [10]:

$$\psi(x) \rightarrow \psi(x) e^{-f(x)}. \quad (2.2)$$

Then,

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + V(x), \quad (2.3)$$

$$H_G = -\frac{1}{2} \frac{d^2}{dx^2} + A(x) \frac{d}{dx} + \Delta V, \quad (2.4)$$

where

$$\Delta V = V(x) + \frac{1}{2} A'(x) - \frac{1}{2} A^2(x) \quad (2.5)$$

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while

$$f(x) = \int^x A(x') dx'. \quad (2.6)$$

The gauge transformed eigenvalue equation reads

$$H_G \tilde{\psi}(x) = E \tilde{\psi}(x). \quad (2.7)$$

Next we consider a finite dimensional representation of the $SU(2)$ group characterised by finite spins. The generators of the group are

$$\begin{aligned} T^+ &= 2j\zeta - \zeta^2 \frac{d}{d\zeta}, \\ T^0 &= -j + \zeta \frac{d}{d\zeta}, \\ T^- &= \frac{d}{d\zeta}. \end{aligned} \quad (2.8)$$

T^+ , T^0 , T^- satisfy the commutation relations

$$[T^+, T^-] = 2T^0, \quad [T^\pm, T^0] = T^\pm. \quad (2.9)$$

The basis of the corresponding finite dimensional representation is

$$R^j = (1, \zeta, \zeta^2, \dots, \zeta^{2j}). \quad (2.10)$$

We choose the gauge in such a way that H_G can be written as

$$H_G = \sum_{a,b \neq 0} C_{ab} T^a T^b + \sum_{a,b \neq 0} C_a T^a + \text{constant}, \quad (2.11)$$

where C_{ab} and C_a are numerical coefficients. Using (2.8), (2.11) can be written as

$$H_G = -\frac{1}{2} P_4(\zeta) \frac{d^2}{d\zeta^2} + P_3(\zeta) \frac{d}{d\zeta} + P_2(\zeta), \quad (2.12)$$

where $P_n(\zeta)$ denotes at most a polynomial of degree n in ζ . To bring (2.12) in a Schrödinger like form, we put say

$$x = \int d\zeta P_4^{-1/2}(\zeta) = F(\zeta). \quad (2.13)$$

Then

$$H_G = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{(1/4)P_4' + P_3}{P_4^{1/2}} \frac{d}{dx} + P_2. \quad (2.14)$$

To get the deformed Hamiltonian H_G^q one uses the generators T_q^\pm , T_q^0 in place of T^\pm , T^0 , respectively, in (2.11). T_q^\pm , T_q^0 are the generators of $SU_q(2)$ satisfying the commutations relations

$$\begin{aligned} [T_q^+, T_q^-] &= \frac{\sinh 2\tau T_q^0}{\sinh \tau}, \\ [T_q^\pm, T_q^0] &= T_q^\pm, \end{aligned} \quad (2.15)$$

where $\tau = \ln q$, q being a real number. In the limit $\tau \rightarrow 0$ (or $q \rightarrow 1$) the deformation disappears.

3. The Non Polynomial Oscillator

Here we take

$$V(r) = r^2 + \frac{\lambda r^2}{1 + g r^2} + \frac{l(l+1)}{\tau^2}. \quad (3.1)$$

We take the gauge function to be

$$A(r) = -r + \frac{2g}{1 + g r^2} + \frac{b}{\tau}. \quad (3.2)$$

In general $b = l + 1$. However, for $l = 0$ we can take both $b = 0$ and $b = 1$, the even parity and odd parity solutions. It can easily be seen that the gauge transformed Hamiltonian is

$$H_G = -\frac{d^2}{dr^2} - 2W(r) \frac{d}{dr} + \Delta V, \quad (3.3)$$

where

$$\Delta V = V(r) - (A^2 + A'). \quad (3.4)$$

Putting $\zeta(\tau) = r^2$, we obtain

$$\begin{aligned} H_G &= -4\zeta \frac{d^2}{d\zeta^2} - 2 \frac{d}{d\zeta} + \left(4\zeta - \frac{8g\zeta}{1+g\zeta} - 4b \right) \frac{d}{d\zeta} \\ &\quad + \frac{\lambda}{g} + (2b+1) - \frac{(4bg+2g+\lambda/g)}{1+g\zeta} + \frac{4g\zeta}{1+g\zeta}. \end{aligned} \quad (3.5)$$

As we can see, H_G can not be immediately written in the form (2.12). However, we can use the following procedure. Define

$$\Omega_G = (1 + g\zeta)(H_G - E). \quad (3.6)$$

Using (3.5) we obtain

$$\begin{aligned} \Omega_G &= -4(\zeta + g\zeta^2) \frac{d^2}{d\zeta^2} \\ &\quad + \{4g\zeta^2 + (4 - 4bg - 10g)\zeta - (4b+2)\} \frac{d}{d\zeta} \\ &\quad + (\lambda/g + 2b + 5 - E)g\zeta + (2b + 1 - E - 4bg - 2g). \end{aligned} \quad (3.7)$$

In terms of T^\pm and T^0 , Ω_G can be written as

$$\begin{aligned} \Omega_G &= AT^{0^2} + DT^-T^0 + FT^-T^+ + GT^+ \\ &\quad + H_\epsilon T^- + IT^0. \end{aligned} \quad (3.8)$$

Using (2.8), we get

$$\Omega_G = \{(A-F)\xi^2 + D\xi\} \frac{d^2}{d\xi^2} + \{(1-2j)A + 2(j-1)F + I)\xi - G\xi^2 + H_c + D(i-j)\} \frac{d}{d\xi} + (2jG\xi + Aj^2 + 2jF - jI). \quad (3.9)$$

Comparison of (3.9) and (3.7) gives

$$\begin{aligned} D &= -4, \quad A-F = -4g, \quad G = -4g, \\ D(i-j) + H_c &= -4b-2, \\ (l-2j)A + 2(j-1)F + I &= 4-8g-4bg-2g, \\ (2b+1)E - 4bg-2g &= Aj^2 + 2jF - jI, \\ E &= \frac{\lambda}{g} + 2b + 8j + 5. \end{aligned} \quad (3.10)$$

Now, to obtain the deformed Schrödinger equation corresponding to Ω_G in (3.9) we keep the coefficients A, D, F etc. unchanged but replace T^\pm, T^0 by T_q^\pm and T_q^0 in (3.8).

We consider the following cases:

(i) $j = 0$.

Equation (3.10) gives

$$\lambda/g = -4bg - 2g - 4, \quad (3.11)$$

so that for $b = 0$,

$$\lambda/g = -2g - 4, \quad E = -2g - 1, \quad (3.12)$$

and for $b = 1$

$$\lambda/g = -6g - 4, \quad E = -6g + 3, \quad (3.13)$$

which agree with the known results.

Also for $j = 0$, the deformation does not lead to any change.

(ii) $j = 1/2$.

Here T^\pm, T^0 are given by

$$T^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad T^0 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}. \quad (3.14)$$

It is found that in this case

$$\frac{\sinh 2\tau T^0}{\sinh \tau} = 2T^0.$$

Hence no deformation occurs and we can take

$$T_q^\pm = T^\pm, \quad T_q^0 = T^0.$$

Substitution of (3.14) in (3.8) gives Ω_G as

$$\Omega_G^q = \Omega_G = \begin{pmatrix} A/4 + I/2 & G \\ D/2 + H_c & A/4 + F - I/2 \end{pmatrix}. \quad (3.15)$$

The eigenvalue equation

$$H_G^q \psi = E_q \psi \quad (3.15a)$$

implies $\det \Omega_G^q = 0$, which leads to

$$(A/4 + I/2)(A/4 + F - I/2) - G(D/2 + H_c) = 0, \quad (3.16)$$

which gives

$$\begin{aligned} \lambda/g &= -(7g + 6bg + 6) \pm \{(7g + 6bg + 6)^2 \\ &\quad - (32b^2g^2 + 64bg^2 + 64bg \\ &\quad + 24g^2 + 96g + 32)\}^{1/2}, \end{aligned} \quad (3.17)$$

so that for $b = 0$

$$\begin{aligned} \lambda/g &= -(7g + 6) \pm \sqrt{25g^2 - 12g + 4}, \\ E &= \lambda/g + 9, \end{aligned} \quad (3.18)$$

and for $b = 1$

$$\begin{aligned} \lambda/g &= -(13g + 6) \pm \sqrt{49g^2 - 4g + 4}, \\ E &= \lambda/g + 11, \end{aligned} \quad (3.19)$$

which agrees with the results of [26].

(iii) $j = 1$.

Here we take

$$\begin{aligned} T^+ &= \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ T^- &= \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ T^0 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \end{aligned} \quad (3.20)$$

It is seen that

$$\frac{\sinh 2\tau T^0}{\sinh \tau} = [2] T^0, \quad (3.21)$$

which allows us to take the following realization of T_q^\pm, T_q^0 :

$$T_q^\pm = T^\pm \left(\frac{[2]}{2} \right)^{1/2}, \quad T_q^0 = T^0. \quad (3.22)$$

Thus the deformed equation for Ω_G takes the form

$$\Omega_G^q = A T^{0^2} + \alpha D T^- T^0 + \alpha^2 F T^- T^+ + \alpha G T^+ + \alpha H_c T^- + I T^0, \quad (3.23)$$

where

$$\alpha = \left(\frac{[2]}{2} \right)^{1/2}. \quad (3.24)$$

Thus

$$\begin{aligned} \Omega_G^q &= \{(A - \alpha^2 F) \xi^2 + \alpha D \xi\} \frac{d^2}{d\xi^2} \\ &+ \{(1 - 2j)A + 2(j - 1)\alpha^2 F + I\} \xi - \alpha G \xi^2 \\ &+ \alpha(D(1 - j) + H_c) \frac{d}{d\xi} + A j^2 + 2j \alpha^2 F - jI + 2j \alpha G \xi. \end{aligned} \quad (3.25)$$

The corresponding deformed Hamiltonian is

$$H_q = -\frac{d^2}{dr^2} + V_q(r), \quad (3.26)$$

where

$$\begin{aligned} V_q(r) &= \frac{\alpha^2 g^2}{g_1^2} r^2 + \frac{b(b-1)}{r^2} + \frac{\lambda \alpha r^2}{\alpha + g_1 r^2} - \frac{\left(\frac{\alpha \beta^2}{g_1} + \alpha \beta\right)}{(\alpha + g_1 r^2)^2} \\ &+ \left(\frac{4\alpha g}{g_1} + \frac{\alpha \beta g}{g_1^2}\right) + \frac{1}{\alpha + g_1 r^2} \left\{ -2g - 4bg - \beta \left(\frac{1}{2} + b\right) \right\} \\ &+ \lambda \alpha^2 \left(\frac{1}{g_1} - \frac{1}{g}\right) - 2(\alpha^2 - 1)(4 + 6g + 4bg) - 12 \\ &\cdot \left(1 - \frac{\alpha^2 g}{g_1}\right) - \frac{\alpha^2}{g_1} \left(4g + \frac{\beta g}{g_1}\right) + \left(\beta + \frac{\beta^2}{4g_1}\right) \Big\}, \end{aligned} \quad (3.27)$$

where

$$g_1 = \frac{1}{4} \{4g + (1 - \alpha^2)(4 + 6g + 4bg + \lambda/2g)\} \quad (3.28)$$

and

$$\beta = 2 - 5g - 2bg + g_1 \quad (3.29)$$

Since $\alpha \rightarrow 1$ as $q \rightarrow 1$,

$$V_q(r) \rightarrow r^2 + \frac{l(l+1)}{r^2} + \frac{\lambda r^2}{1 + g r^2} = V(r) \quad (3.30)$$

and

$$E_q \rightarrow E \quad \text{as } q \rightarrow 1.$$

λ, g satisfy the constraint given by

$$\det(\Omega_G^q) = 0, \quad (3.31)$$

where

$$\Omega_G^q = \begin{pmatrix} A + I & [2]^{1/2} G & 0 \\ [2]^{1/2} (D + H_c) & [2] F & [2]^{1/2} G \\ 0 & [2]^{1/2} H_c & A - I + [2] F \end{pmatrix}. \quad (3.32)$$

However, if we fix both λ and g , a simplification results: Take

$$A + I = 0, \quad (3.33)$$

then (3.3) reduces (for $G \neq 0$) to

$$(D + H_c)(A - I + [2] F) = 0 \quad (3.34)$$

or,

$$\begin{aligned} &-12(1 + [2]) - 8b(1 + [2]) + (32 - 12[2])bg \\ &+ (30 - 18[2])g + 8b^2g - \frac{[2]\lambda b}{g} - \frac{3[2]}{2} \frac{\lambda}{g} = 0. \end{aligned} \quad (3.35)$$

Also (3.33) gives

$$4 + 30g + \frac{\lambda}{g} + 4bg = 0. \quad (3.36)$$

From (3.35) and (3.36), we get

$$g = \frac{(12 + 8b) + [2](6 + 4b)}{(30 + 32b + 8b^2) + [2]\{27 + 24b + 4b^2\}}, \quad (3.37)$$

$$[2] = \sinh 2\tau / \sinh \tau \quad (3.38)$$

$$\lambda/g = -\{4 + (30 + 4b)g\}, \quad (3.39)$$

and

$$E_q = \frac{\alpha g}{g_1} (\lambda/g + 2b + 13). \quad (3.40)$$

In particular we consider the following cases:

(i) $b = 0$. Keeping terms of $0(\tau^2)$ only, we get

$$g \simeq \frac{2}{7} (1 - \tau^2/14), \quad (3.41)$$

$$\frac{\lambda}{g} \simeq -\frac{88}{7} + \frac{30}{49} \tau^2, \quad (3.42)$$

$$E_q \simeq \frac{3}{7} + \frac{30}{49} \tau^2. \quad (3.43)$$

(ii) $b = 1$. Here also we keep terms of $0(\tau^2)$. g and λ/g are given by

$$g \simeq \frac{2}{9} \left(1 - \frac{1}{18} \tau^2\right), \quad (3.44)$$

$$\frac{\lambda}{g} \simeq -\frac{104}{9} + \frac{34}{81} \tau^2, \quad (3.45)$$

$$E_q \simeq \frac{31}{9} + \frac{59}{126} \tau^2. \quad (3.46)$$

As $\tau \rightarrow 0$, the results agree completely with those obtained in [26].

4. Conclusion and Discussion

In this paper we have formulated a scheme by which the deformed wave equations and exact solutions of quasi-exactly solvable potentials can be obtained, using the finite dimensional representation of $SU_q(2)$. For a particular case the non-polynomial potential $V(r) = r^2 + \frac{\lambda r^2}{1 + g r^2}$ was studied.

It has been shown that for the $j=0$ and $j=1/2$ representation of $SU(2)$ no deformation occurs. However, for $j=1$, deformation occurs and one gets exact solution if λ and g satisfy certain q dependent constraint relations. The deformed potential turns out to be qualitatively different from the undeformed one, as the former has an additional term of the form

$\frac{1}{(1 + g_1 r^2)^2}$. Though we confined ourselves upto $j=1$ representations, the scheme would work for any j . Only in cases $j > 1$ a simple relation between T_q^\pm and T^\pm can not be obtained. For example for $j=1\frac{1}{2}$, we can take the following representation of T_q^\pm, T_q^0 :

$$T_q^\pm = \begin{pmatrix} 0 & \sqrt{[3]} & 0 & 0 \\ 0 & 0 & [2] & 0 \\ 0 & 0 & 0 & \sqrt{[3]} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T_q^- = (T_q^+)^+, \text{ and}$$

$$T_q^0 = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix},$$

One can then obtain the constraint relation using (3.31) and the corresponding energy from (3.10).

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